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An upper bound on the Perron value of an almost regular tournament matrix[☆]

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Abstract

We provide an asymptotic upper bound on the Perron value of an almost regular tournament matrix, improving upon an existing bound due to Friedland. Our approach employs techniques from constrained optimization.

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1. Introduction and background

A *tournament* on vertices $1, \dots, n$ is a loop-free directed graph D with the property that for each pair of distinct vertices i and j , D contains exactly one of the arcs $i \rightarrow j$ and $j \rightarrow i$. A *tournament matrix* is the $(0, 1)$ adjacency matrix of a tournament, or equivalently, a $(0, 1)$ matrix T such that

$$T + T^t = J - I, \quad (1)$$

where J denotes the all ones matrix. There is a wealth of literature on tournaments ([1,12] provide surveys of some older results) and the last decade has seen an emerging body of work on tournament matrices.

Since a tournament matrix T is an entrywise nonnegative matrix, Perron–Frobenius theory applies (see [2]), so the spectral radius of T , $\rho(T)$, is an

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eigenvalue. In the event that T is irreducible (equivalently, if the corresponding tournament is strongly connected), we refer to ρ as the *Perron value* for T ; in that case, the corresponding eigenvector can be taken to have all positive entries, and is known as a *Perron vector* for T . A result of Brauer and Gentry [3] asserts that if T is an $n \times n$ tournament matrix, then $\rho(T) \leq (n-1)/2$, with equality holding if and only if T is a *regular* tournament matrix, i.e., $T\mathbf{1} = ((n-1)/2)\mathbf{1}$, where $\mathbf{1}$ denotes the all ones vector. Observe that if T is a regular tournament matrix of order n , then necessarily n is odd, since $(n-1)/2$ must be an integer in that case; we note that regular tournament matrices are known to exist in all odd orders. Consequently, for odd values of n , $(n-1)/2$ provides an attainable upper bound on the spectral radius of any tournament matrix of order n .

This immediately raises the problem of finding an attainable upper bound on the spectral radius of a tournament matrix of even order. Indeed, it suffices to find an attainable upper bound on the Perron value of an irreducible tournament matrix of even order. In 1983, Brualdi and Li [4] conjectured that for each even n , the tournament matrix of order n which maximizes the Perron value can be written as

$$B_n = \left[\begin{array}{c|c} U_{n/2} & U_{n/2}^t \\ \hline U_{n/2}^t + I & U_{n/2} \end{array} \right],$$

where $U_{n/2}$ denotes the matrix of order $n/2$ with ones above the diagonal, and zeros on and below the diagonal. Observe that $U_{n/2}$ is itself a tournament matrix; the corresponding tournament is known as a *transitive tournament*.

While Brualdi and Li's conjecture is still open, there has been some progress made on it. The matrix B_n is an example of an *almost regular* tournament matrix, i.e., a tournament matrix having $n/2$ row sums equal to $(n-2)/2$ and $n/2$ row sums equal to $n/2$. It is shown in [9] that for sufficiently large even n , an $n \times n$ tournament matrix which maximizes the Perron value must be almost regular. The main result of Kirkland [8] asserts that

$$\rho(B_n) = \frac{n-1}{2} - \frac{e^2-1}{2(e^2+1)n} + \mathcal{O}(1/n^3)$$

where as usual, $\mathcal{O}(1/n^k)$ denotes a sequence a_n such that the sequence $n^k a_n$ is bounded. A result of Friedland [5] shows that for any almost regular tournament matrix T of order n

$$\rho(T) \leq \frac{n-1}{2} - \frac{3}{8n} + \mathcal{O}(1/n^2). \quad (2)$$

Pulling these conclusions together, we see that for all sufficiently large even n , a tournament matrix T which maximizes the Perron value satisfies

$$\rho(T) = \frac{n-1}{2} - \frac{\gamma_n}{n} + \mathcal{O}(1/n^2),$$

where

$$0.375 = \frac{3}{8} \leq \gamma_n \leq \frac{e^2-1}{2(e^2+1)} \approx 0.380797 \dots \quad (3)$$

Our goal in the present paper is to narrow the gap between these upper and lower bounds on γ_n by improving the lower bound of $\frac{3}{8}$ (equivalently, by sharpening Friedland's result). To do so, we exploit some of the properties of the Perron vector for an almost regular tournament matrix. Those properties are developed in Section 2. We remark that throughout this paper, we are thinking of n as a large even integer, for that is the context in which our results have the most significance.

2. Main results

Suppose that T is an almost regular tournament matrix of order n with Perron value ρ ; without loss of generality we assume that the first $n/2$ rows of T sum to $(n-2)/2$ and the last $n/2$ rows of T sum to $n/2$. Partition T as

$$T = \left[\begin{array}{c|c} T_1 & S \\ \hline J - S^t & T_2 \end{array} \right], \quad (4)$$

where each block is $n/2 \times n/2$. Let $T_1 \mathbf{1} = \mu$ and $T_2 \mathbf{1} = \nu$.

For any tournament matrix M , its vector of row sums $M\mathbf{1}$ is known as the *score vector* for that matrix (in particular, μ and ν above are the score vectors for T_1 and T_2 , respectively). It follows readily from (1) that if M is of order k and has score vector σ , then the tournament matrix M^t has score vector $(k-1)\mathbf{1} - \sigma$. A result of Landau [11], which will be needed in the sequel, gives a criterion for a vector of nonnegative integers to be the score vector of some tournament matrix: Let σ be a vector of integers

$$\begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_k \end{bmatrix},$$

where without loss of generality we take $\sigma_1 \leq \dots \leq \sigma_k$. Then σ is the score vector of some $k \times k$ tournament matrix if and only if

$$\sum_{i=1}^j \sigma_i \geq j(j-1)/2 \text{ for } j = 1, \dots, k-1 \quad \text{and} \quad \sum_{i=1}^k \sigma_i = k(k-1)/2.$$

Throughout, the score vector

$$\begin{bmatrix} 0 \\ 1 \\ \vdots \\ \frac{n-2}{2} \end{bmatrix},$$

which corresponds to a transitive tournament on $n/2$ vertices, will be denoted by τ . We note that the only score vectors without repeated entries correspond to transitive tournaments (see [12]).

Consider the right Perron vector w of our almost regular tournament matrix T of (4), and partition w conformally with T as

$$w = \begin{bmatrix} u \\ v \end{bmatrix};$$

we take w to be normalized so that $u^t \mathbf{1} + v^t \mathbf{1} = n$. Note that

$$\rho n = \rho(u^t \mathbf{1} + v^t \mathbf{1}) = \rho \mathbf{1}^t w = \mathbf{1}^t T w = \left[\frac{n-2}{2} \mathbf{1}^t \mid \frac{n-2}{2} \mathbf{1}^t \right] \begin{bmatrix} u \\ v \end{bmatrix}.$$

It follows that $(n/2 - \rho) \mathbf{1}^t u = (\rho - (n-2)/2) \mathbf{1}^t v$, and we deduce that

$$\mathbf{1}^t u = n(\rho - (n-2)/2) \quad (5)$$

and that

$$\mathbf{1}^t v = n(n/2 - \rho). \quad (6)$$

Further

$$\rho \mathbf{1}^t u = [\mathbf{1}^t | \mathbf{0}^t] T w = [\mathbf{1}^t T_1 | \mathbf{1}^t S] w = \left[\frac{n-2}{2} \mathbf{1}^t - \mu^t | v^t \right] \begin{bmatrix} u \\ v \end{bmatrix},$$

and we find that

$$-\mu^t u + v^t v = n(\rho - (n-2)/2)^2. \quad (7)$$

A result of Kirkland [10] yields two more useful conditions on u and v . In [10] it is shown (in the language of the present paper) that for each $i, j = 1, \dots, n/2$, $u_i < v_j$. From the eigenvalue–eigenvector equation, we have $\rho v_j = e_{j+n/2}^t T w$, so that ρv_j is the sum of $n/2$ entries of w . Since each entry of u is less than each entry of v , we find that $e_{j+n/2}^t T w \geq \mathbf{1}^t u$, which yields, upon using (5)

$$v_j \geq (n/\rho)(\rho - (n-2)/2) \equiv v_{\min}, \quad 1 \leq j \leq n/2. \quad (8)$$

Similarly, we have $\rho u_i = e_i^t T w \leq \mathbf{1}^t v - v_{\min}$, yielding

$$u_i \leq (n/\rho^2)((n-2)/2 + (n-2)/2\rho - \rho^2) \equiv u_{\max}, \quad 1 \leq i \leq n/2. \quad (9)$$

Much of our work in the sequel is devoted to producing a lower bound on $u^t u + v^t v$ for any pair of vectors u, v satisfying constraints (5)–(9), using the techniques of constrained optimization. In order to do so, it is notationally convenient to deal not with the vectors u and v , but rather with the related vectors x and y defined as $x \equiv u_{\max} \mathbf{1} - u$ and $y \equiv v - v_{\min} \mathbf{1}$. Observe that the constraints (9), (8), (5), (6) and (7) on u and v can be recast in terms of x and y as follows:

$$x \geq \mathbf{0}, \quad y \geq \mathbf{0}; \quad (10)$$

$$x^t \mathbf{1} = \frac{n}{2} u_{\max} - \mathbf{1}^t u = \frac{(\rho+1)n}{\rho} (n(n-2)/4 - \rho^2); \quad (11)$$

$$y^t \mathbf{1} = \mathbf{1}^t v - \frac{n}{2} v_{\min} = \frac{n}{\rho} (n(n-2)/4 - \rho^2); \quad (12)$$

$$\begin{aligned} \mu^t x + v^t y &= n(\rho - (n-2)/2)^2 - \frac{n^2(n-2)}{8\rho^2} \\ &\quad \times (2\rho^2 - (n-2)\rho - (n-2)/2). \end{aligned} \quad (13)$$

Given two score vectors μ and ν , each of which has $n/2$ entries, and a positive number ρ , we denote the set of admissible pairs of vectors x, y as follows:

$$\mathcal{A}(\mu, \nu, \rho) \equiv \left\{ (x, y) \mid x, y \geq \mathbf{0}; \ x^t \mathbf{1} = \frac{(\rho+1)n}{\rho} (n(n-2)/4 - \rho^2); \right. \\ \left. y^t \mathbf{1} = \frac{n}{\rho} (n(n-2)/4 - \rho^2); \ \mu^t x + \nu^t y = n(\rho - (n-2)/2)^2 \right. \\ \left. - \frac{n^2(n-2)}{8\rho^2} (2\rho^2 - (n-2)\rho - (n-2)/2) \right\}.$$

Whenever $\mathcal{A}(\mu, \nu, \rho) \neq \emptyset$, we also define $m(\mu, \nu, \rho)$ as

$$m(\mu, \nu, \rho) \equiv \min\{x^t x + y^t y \mid (x, y) \in \mathcal{A}(\mu, \nu, \rho)\}.$$

The following will lead us to the point where we can focus our attention on the case that $\mu = \nu = \tau$.

Lemma 2.1. *Suppose that n is even, and let μ and ν be score vectors having $n/2$ entries. Let ρ be a positive number such that $\mathcal{A}(\mu, \nu, \rho) \neq \emptyset$. Suppose that we have a pair of vectors $(x, y) \in \mathcal{A}(\mu, \nu, \rho)$ such that $x^t x + y^t y = m(\mu, \nu, \rho)$. If $\mu_i = \mu_j$, then $x_i = x_j$. Similarly, if $\nu_i = \nu_j$, then $y_i = y_j$.*

Proof. Suppose to the contrary that $\mu_i = \mu_j$ but $x_i < x_j$. Form \hat{x} from x by replacing both x_i and x_j by $(x_i + x_j)/2$. Then $\hat{x}^t \mathbf{1} = x^t \mathbf{1}$ and $\mu^t \hat{x} = \mu^t x$, so we find that $(\hat{x}, y) \in \mathcal{A}(\mu, \nu, \rho)$. But $\hat{x}^t \hat{x} + y^t y < x^t x + y^t y$, which contradicts the minimality of $m(\mu, \nu, \rho)$. The statement concerning ν and y is proved similarly. \square

Corollary 2.2. *Suppose that n, μ, ν and ρ are as in Lemma 2.1. Then $\mathcal{A}(\tau, \tau, \rho) \neq \emptyset$ and $m(\tau, \tau, \rho) \leq m(\mu, \nu, \rho)$.*

Proof. Without loss of generality, we suppose that the entries of μ and ν are non-decreasing. We claim that $\mathcal{A}(\tau, \nu, \rho) \neq \emptyset$ and that $m(\tau, \nu, \rho) \leq m(\mu, \nu, \rho)$. We establish this claim by induction on the quantity

$$d(\mu) \equiv \sum_{p=1}^{n/2} \left(\sum_{l=1}^p \mu_l - p(p-1)/2 \right),$$

which, by Landau's criterion, is a nonnegative integer. If $d(\mu) = 0$, it follows readily that $\mu = \tau$, and the claim certainly holds. If $d(\mu) \geq 1$, then $\mu \neq \tau$, and so μ has some repeated entries. Let k be the first index such that for some $j \geq 2$ we have $\mu_k < \mu_{k+1} = \dots = \mu_{k+j} < \mu_{k+j+1}$ (if $\mu_1 = \mu_2$, we take $k = 0$ and if $\mu_{k+1} = \mu_{n/2}$, we take $j = n/2 - k$); let a be the common value of $\mu_{k+1}, \dots, \mu_{k+j}$.

Suppose that $k \geq 1$. Now let $\hat{\mu}$ be the vector formed from μ by replacing μ_{k+1} by $a - 1$ and μ_{k+j} by $a + 1$, and note that the entries of $\hat{\mu}$ are nondecreasing. Observe that for $p = 1, \dots, k$ and $p = k + j + 1, \dots, n/2$ we have

$$\sum_{l=1}^p \hat{\mu}_l = \sum_{l=1}^p \mu_l \geq p(p-1)/2,$$

the inequality following from Landau's criterion. Further, for each $1 \leq i \leq j-1$, we have

$$\sum_{l=1}^{k+i} \hat{\mu}_l = \left(\sum_{l=1}^{k+i} \mu_l \right) - 1.$$

Suppose that for some $1 \leq i \leq j-1$,

$$\sum_{l=1}^{k+i} \hat{\mu}_l < (k+i)(k+i-1)/2;$$

then necessarily

$$\sum_{l=1}^{k+i} \mu_l = (k+i)(k+i-1)/2,$$

so that

$$(k+i)(k+i-1)/2 = \sum_{l=1}^{k+i} \mu_l \geq k(k-1)/2 + ia.$$

It now follows that $a \leq (2k-1+i)/2$. But then we find that

$$\begin{aligned} \sum_{l=1}^{k+i+1} \mu_l &= \sum_{l=1}^{k+i} \mu_l + a \leq (k+i)(k+i-1)/2 + (2k-1+i)/2 \\ &= (k+i+1)(k+i)/2 - (i+1)/2, \end{aligned}$$

contradicting the fact that μ satisfies Landau's criterion. A similar argument applies if $k=0$, and we find that in either case, $\hat{\mu}$ is the score vector for some tournament matrix of order $n/2$. Note also that $d(\hat{\mu}) = d(\mu) - j + 1$.

Select vectors x and y so that $(x, y) \in \mathcal{A}(\mu, v, \rho)$ and $x^t x + y^t y = m(\mu, v, \rho)$. From Lemma 2.1, we have $x_{k+1} = \dots = x_{k+j}$, from which we find that $x^t \hat{\mu} = x^t \mu$. Thus $(x, y) \in \mathcal{A}(\hat{\mu}, v, \rho)$, and we also have $m(\hat{\mu}, v, \rho) \leq x^t x + y^t y = m(\mu, v, \rho)$. Since $d(\hat{\mu}) < d(\mu)$, an application of the induction hypothesis shows that $\mathcal{A}(\tau, v, \rho) \neq \emptyset$ and that $m(\tau, v, \rho) \leq m(\hat{\mu}, v, \rho)$. Applying the inequality $m(\hat{\mu}, v, \rho) \leq m(\mu, v, \rho)$ completes the proof of the claim.

An analogous argument applies to v , and the conclusions follow. \square

The preceding result directs our attention towards $m(\tau, \tau, \rho)$. For a certain range of values of ρ , the next result gives an expression for $m(\tau, \tau, \rho)$ which is accurate to terms in $1/n$.

Theorem 2.3. Suppose that n is even, that $\rho = (n - 1)/2 - \gamma/n + \mathcal{O}(1/n^2)$, where $1/2 > \gamma > 1/3$, and that $\mathcal{A}(\tau, \tau, \rho) \neq \emptyset$. Then

$$m(\tau, \tau, \rho) = \frac{4(4\gamma - 1)^3}{(27\gamma - 9)n} + \mathcal{O}(1/n^2).$$

Proof. We begin by denoting

$$b_1 \equiv \frac{(\rho + 1)n}{\rho} (n(n - 2)/4 - \rho^2), \quad b_2 \equiv \frac{n}{\rho} (n(n - 2)/4 - \rho^2)$$

and

$$b_3 \equiv n(\rho - (n - 2)/2)^2 - \frac{n^2(n - 2)}{8\rho^2} (2\rho^2 - (n - 2)\rho - (n - 2)/2).$$

We find that

$$b_1 = 2(\gamma - 1/4) + \mathcal{O}(1/n), \quad b_2 = 2(\gamma - 1/4) + \mathcal{O}(1/n)$$

and $b_3 = n\gamma/2 + \mathcal{O}(1)$.

We seek to minimize the quantity $x^t x + y^t y$ subject to $(x, y) \in \mathcal{A}(\tau, \tau, \rho)$. Evidently this is a quadratic programming problem where the function to be minimized is a positive definite quadratic form, and where the constraints consist of nonnegativity, and some linear equalities. According to Kuhn–Tucker theory (see [6], for example), this quadratic programming problem attains its absolute minimum at a (unique) pair x, y satisfying the following conditions:

- (a) $(x, y) \in \mathcal{A}(\tau, \tau, \rho)$ — i.e., $x, y \geq 0$, $x^t \mathbf{1} = b_1$, $y^t \mathbf{1} = b_2$ and $\tau^t x + \tau^t y = b_3$.
- (b) There are vectors $\alpha, \beta \geq 0$ and scalars a_1, a_2, a_3 such that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \tau \\ \mathbf{0} & \mathbf{1} & \tau \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

- (c) For $i = 1, \dots, n/2$ we have $x_i \alpha_i = 0$ and $y_i \beta_i = 0$.

In particular we see that for each $i, j = 1, \dots, n/2$, $x_i = a_1 + a_3(i - 1) + \alpha_i$ and $y_j = a_2 + a_3(j - 1) + \beta_j$. Applying (c), we find that if $x_i > 0$, then necessarily $\alpha_i = 0$, so that $a_1 + a_3(i - 1) > 0$, while if $x_i = 0$ then we have $a_1 + a_3(i - 1) = -\alpha_i \leq 0$. Thus we see that

$$x_i \text{ is positive or zero according as } a_1 + a_3(i - 1) \text{ is positive or nonpositive,} \\ \text{respectively.} \quad (14)$$

Similarly,

$$y_j \text{ is positive or zero according as } a_2 + a_3(j - 1) \text{ is positive or nonpositive,} \\ \text{respectively.} \quad (15)$$

Suppose first that $a_3 \leq 0$. In that case, if $x_i = 0$ for some i , then from (14) we see that necessarily $x_{i+1} = 0$. A similar argument applies if $y_j = 0$ for some j , and we find that $\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}$ can be written as

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \tilde{x} \\ \mathbf{0} \\ \tilde{y} \\ \mathbf{0} \end{bmatrix},$$

where for some integers k and l we have

$$\tilde{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} > \mathbf{0} \quad \text{and} \quad \tilde{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_l \end{bmatrix} > \mathbf{0}.$$

From (b) and (c) we have

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \mathbf{1}_k & \mathbf{0}_k & \tau_k \\ \mathbf{0}_l & \mathbf{1}_l & \tau_l \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

where the subscripts on $\mathbf{1}$ and $\mathbf{0}$ denote their orders, and where for any $1 \leq p \leq n/2$, τ_p is the vector consisting of the first p entries of τ . Applying the constraints on $x^t \mathbf{1}$, $y^t \mathbf{1}$ and $\tau^t x + \tau^t y$, it follows that

$$\begin{bmatrix} \mathbf{1}_k^t & \mathbf{0}_l^t \\ \mathbf{0}_k^t & \mathbf{1}_l^t \\ \tau_k^t & \tau_l^t \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} k & 0 & \frac{k(k-1)}{2} \\ 0 & l & \frac{l(l-1)}{2} \\ \frac{k(k-1)}{2} & \frac{l(l-1)}{2} & \left(\frac{k(k-1)(2k-1)}{6} + \frac{l(l-1)(2l-1)}{6} \right) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

A couple of row operations reveal that

$$\begin{aligned} & \left(\frac{k(k-1)(k+1)}{12} + \frac{l(l-1)(l+1)}{12} \right) a_3 \\ &= b_3 - \frac{k-1}{2} b_1 - \frac{l-1}{2} b_2 \\ &\geq b_3 - \frac{n-2}{4} (b_1 + b_2) \\ &= n\gamma/2 - (n/2)(2\gamma - 1/2) + \mathcal{O}(1) \\ &= (1/2 - \gamma)n/2 + \mathcal{O}(1) > 0, \end{aligned}$$

contradicting our assumption that $a_3 \leq 0$. We thus conclude that $a_3 > 0$.

Since $a_3 > 0$, we see from (14) that if $x_i = 0$ for some i , then necessarily $x_{i-1} = 0$. A similar argument applies if $y_j = 0$ for some j , and we conclude that $\begin{bmatrix} x \\ y \end{bmatrix}$ can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \tilde{x} \\ \mathbf{0} \\ \tilde{y} \end{bmatrix},$$

where for some integers k and l we have

$$\tilde{x} = \begin{bmatrix} x_{n/2-k+1} \\ \vdots \\ x_{n/2} \end{bmatrix} > \mathbf{0} \quad \text{and} \quad \tilde{y} = \begin{bmatrix} y_{n/2-l+1} \\ \vdots \\ y_{n/2} \end{bmatrix} > \mathbf{0}.$$

Observe that the case $k = l = 1$ is impossible, otherwise we would have $x_{n/2} = b_1$, $y_{n/2} = b_2$ (from (11) and (12)) and hence $(b_1 + b_2)(n - 2)/2 = b_3$ (from (13)); this last reduces to $(2\gamma - 1/2)(n - 2) = \gamma n/2 + \mathcal{O}(1)$, a contradiction since $\gamma > 1/3$. Consequently, at least one of k and l is greater than 1. Arguing as above, we find that

$$\begin{bmatrix} k & 0 & \frac{k(n-k-1)}{2} \\ 0 & l & \frac{l(n-l-1)}{2} \\ \frac{k(n-k-1)}{2} & \frac{l(n-l-1)}{2} & \left(\frac{k(k+1)(k-1)}{12} + \frac{k(n-k-1)^2}{4} + \frac{l(l+1)(l-1)}{12} + \frac{l(n-l-1)^2}{4} \right) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \\ = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

We note that the determinant of the coefficient matrix for this system is

$$kl \left(\frac{k(k-1)(k+1)}{12} + \frac{l(l-1)(l+1)}{12} \right) \equiv kl\Delta,$$

which is positive, since at least one of k and l is at least 2. Solving the system, we find:

$$\begin{aligned} a_1 &= \left(\frac{1}{k} + \frac{(n-k-1)^2}{4\Delta} \right) b_1 + \left(\frac{(n-k-1)(n-l-1)}{4\Delta} \right) b_2 \\ &\quad - \left(\frac{n-k-1}{2\Delta} \right) b_3; \\ a_2 &= \left(\frac{(n-k-1)(n-l-1)}{4\Delta} \right) b_1 + \left(\frac{1}{l} + \frac{(n-l-1)^2}{4\Delta} \right) b_2 \\ &\quad - \left(\frac{n-l-1}{2\Delta} \right) b_3; \\ a_3 &= - \left(\frac{n-k-1}{2\Delta} \right) b_1 - \left(\frac{n-l-1}{2\Delta} \right) b_2 + \frac{1}{\Delta} b_3. \end{aligned}$$

Next we consider the parameters k and l . Note that $x_{n/2-k} = 0 < x_{n/2-k+1}$, and $y_{n/2-l} = 0 < y_{n/2-l+1}$. In light of (14) and (15), these conditions are equivalent to

$$a_1 + a_3(n/2 - k - 1) \leq 0 < a_1 + a_3(n/2 - k)$$

and

$$a_2 + a_3(n/2 - l - 1) \leq 0 < a_2 + a_3(n/2 - l).$$

Using the facts that

$$a_1 = b_1/k - a_3(n - k - 1)/2 \quad (16)$$

and

$$a_2 = b_2/l - a_3(n - l - 1)/2, \quad (17)$$

it follows that our conditions on k and l can be written as

$$\frac{2b_1}{k(k+1)} \leq a_3 < \frac{2b_1}{k(k-1)}$$

and

$$\frac{2b_2}{l(l+1)} \leq a_3 < \frac{2b_2}{l(l-1)}.$$

Since $b_1 = b_2(\rho + 1)/\rho$, we find that necessarily

$$\frac{\rho + 1}{\rho} \frac{1}{k(k-1)} \geq \frac{1}{l(l+1)} \quad \text{and} \quad \frac{1}{l(l-1)} \geq \frac{\rho + 1}{\rho} \frac{1}{k(k+1)}.$$

From these inequalities, we deduce that l is either k or $k - 1$.

Next, we claim that

$$\frac{k}{n} = \frac{3(3\gamma - 1)}{8\gamma - 2} + \mathcal{O}(1/n). \quad (18)$$

To see this, note that we have

$$\begin{aligned} & \frac{2b_1}{k(k+1)} \left(\frac{k(k+1)(k-1)}{12n} + \frac{l(l+1)(l-1)}{12n} \right) \\ & \leq -\frac{(n-k-1)b_1}{2n} - \frac{(n-l-1)b_2}{2n} + \frac{b_3}{n} \\ & < \frac{2b_1}{k(k-1)} \left(\frac{k(k+1)(k-1)}{12n} + \frac{l(l+1)(l-1)}{12n} \right). \end{aligned}$$

Since l is either k or $k - 1$ and $b_2 = b_1 + \mathcal{O}(1/n)$, it follows that

$$-\frac{(n-k)b_1}{n} + \frac{b_3}{n} = \frac{b_1 k}{3n} + \mathcal{O}(1/n),$$

which yields

$$\frac{k}{n} \frac{2b_1}{3} = b_1 - \frac{b_3}{n} + \mathcal{O}(1/n).$$

Since $b_1 = 2(\gamma - 1/4) + \mathcal{O}(1/n)$ and $b_3/n = \gamma/2 + \mathcal{O}(1/n)$, it follows that

$$\frac{k}{n} = \frac{3(3\gamma - 1)}{8\gamma - 2} + \mathcal{O}(1/n),$$

as claimed. Note that necessarily, we also have

$$\frac{l}{n} = \frac{3(3\gamma - 1)}{8\gamma - 2} + \mathcal{O}(1/n).$$

Now observe that

$$\begin{aligned} x^t x &= \tilde{x}^t \tilde{x} \\ &= k a_1^2 + 2 a_1 a_3 \frac{k(n - k - 1)}{2} \\ &\quad + a_3^2 \left(\frac{k n^2}{4} - \frac{n k(k + 1)}{2} + \frac{k(k + 1)(2k + 1)}{6} \right) \\ &= k \left(a_1 + \frac{(n - k - 1)a_3}{2} \right)^2 + a_3^2 \left(\frac{k(k + 1)(k - 1)}{12} \right). \end{aligned}$$

Using (16) and the fact that

$$a_3 = \frac{2b_1}{k^2} + \mathcal{O}(1/n^2),$$

we find that $x^t x = 4b_1^2/3k + \mathcal{O}(1/n^2)$. Finally, using (18) and noting that $b_1 = 2(\gamma - 1/4) + \mathcal{O}(1/n)$, we see that

$$x^t x = \frac{2(4\gamma - 1)^3}{(27\gamma - 9)n} + \mathcal{O}(1/n^2).$$

A similar analysis holds for $y^t y$, yielding

$$x^t x + y^t y = \frac{4(4\gamma - 1)^3}{(27\gamma - 9)n} + \mathcal{O}(1/n^2). \quad \square$$

Corollary 2.4. Suppose that n is even, that μ and v are score vectors of order $n/2$, and that $\rho = (n - 1)/2 - \gamma/n + \mathcal{O}(1/n^2)$ with $1/2 > \gamma > 1/3$. If $\mathcal{A}(\mu, v, \rho) \neq \emptyset$, then

$$m(\mu, v, \rho) \geq \frac{4(4\gamma - 1)^3}{(27\gamma - 9)n} + \mathcal{O}(1/n^2).$$

Proof. From Corollary 2.2 we find that $\mathcal{A}(\tau, \tau, \rho) \neq \emptyset$ and that $m(\mu, v, \rho) \geq m(\tau, \tau, \rho)$. An application of Theorem 2.3 now yields the result. \square

The following will be useful in the proof of the main result.

Lemma 2.5. *Suppose that n is even, and that*

$$\rho = \frac{n-1}{2} - \frac{\gamma}{n} + \mathcal{O}(1/n^2),$$

where $1/2 > \gamma > 1/3$.

$$(a) \quad \frac{n}{2}(u_{\max}^2 + v_{\min}^2) = n + \frac{4\gamma^2 + 4\gamma - 1}{n} + \mathcal{O}(1/n^2).$$

$$(b) \quad v_{\min} - \frac{\rho+1}{\rho}u_{\max} = \frac{-4\gamma}{n} + \mathcal{O}(1/n^2).$$

Proof. Let

$$\theta = n\left(\frac{n-1}{2} - \rho\right), \quad \text{so that } \rho = \frac{n-1}{2} - \frac{\theta}{n}.$$

Observe that $\theta = \gamma + \mathcal{O}(1/n)$.

We begin by giving some expressions for v_{\min} and u_{\max} . We have

$$\begin{aligned} v_{\min} &= (\rho - (n-2)/2)(n/\rho) \\ &= (1/2 - \theta/n) \frac{2n}{n-1-2\theta/n} \\ &= (1-2\theta/n) \frac{1}{1-1/n-2\theta/n^2} \\ &= (1-2\theta/n)(1+1/n+(2\theta+1)/n^2 + \mathcal{O}(1/n^3)) \\ &= 1 + \frac{1-2\theta}{n} + \frac{1}{n^2} + \mathcal{O}(1/n^3). \end{aligned}$$

Also, we have

$$\begin{aligned} u_{\max} &= ((n-2)/2 + \rho(n-2)/2 - \rho^2)n/\rho^2 \\ &= \frac{n}{\rho^2} \left(\frac{n-2}{2} \left(\frac{n+1}{2} - \frac{\theta}{n} \right) - \frac{(n-1)^2}{4} + \frac{(n-1)\theta}{n} - \frac{\theta^2}{n^2} \right) \\ &= \frac{n}{\rho^2} \left(\frac{n-3}{4} + \frac{\theta}{2} - \frac{\theta^2}{n^2} \right) \\ &= \frac{n(n-3+2\theta-4\theta^2/n^2)}{n^2-2n+1-4\theta+4\theta/n+4\theta^2/n^2} \\ &= \left(1 + \frac{2\theta-3}{n} \right) \frac{1}{1-2/n+(1-4\theta)/n^2} + \mathcal{O}(1/n^3) \end{aligned}$$

$$\begin{aligned}
&= \left(1 + \frac{2\theta - 3}{n}\right) \left(1 + \frac{2}{n} + \frac{4\theta + 3}{n^2}\right) + \mathcal{O}(1/n^3) \\
&= 1 + \frac{(2\theta - 1)}{n} + \frac{(8\theta - 3)}{n^2} + \mathcal{O}(1/n^3).
\end{aligned}$$

Consequently,

$$\begin{aligned}
u_{\max}^2 + v_{\min}^2 &= 1 + 2\frac{(2\theta - 1)}{n} + \frac{(2\theta - 1)^2}{n^2} + 2\frac{(8\theta - 3)}{n^2} + 1 \\
&\quad + 2\frac{(1 - 2\theta)}{n} + \frac{(2\theta - 1)^2}{n^2} + \frac{2}{n^2} + \mathcal{O}(1/n^3).
\end{aligned}$$

Thus we find that

$$\frac{n}{2}(u_{\max}^2 + v_{\min}^2) = n + \frac{4\theta^2 + 4\theta - 1}{n} + \mathcal{O}(1/n^2),$$

and recalling that $\theta = \gamma + \mathcal{O}(1/n)$, we see that

$$\frac{n}{2}(u_{\max}^2 + v_{\min}^2) = n + \frac{4\gamma^2 + 4\gamma - 1}{n} + \mathcal{O}(1/n^2).$$

Further,

$$v_{\min} - \frac{\rho + 1}{\rho} u_{\max} = v_{\min} - u_{\max} - 2/n + \mathcal{O}(1/n^2) = \frac{-4\theta}{n} + \mathcal{O}(1/n^2),$$

and (b) now follows from the fact that $\theta = \gamma + \mathcal{O}(1/n)$. \square

Here is our main result.

Theorem 2.6. *Let T be an almost regular tournament matrix of order n with Perron value $\rho = (n - 1)/2 - \gamma/n + \mathcal{O}(1/n^2)$. Then*

$$\gamma \geq \frac{2(3^{2/3}) - 3^{4/3} + 13}{34} \approx 0.377453 \dots$$

Proof. We are done if $\gamma \geq 1/2$, and Friedland's bound (2) shows that $\gamma \geq 3/8$, so we assume henceforth that $1/2 > \gamma > 1/3$. Without loss of generality, we assume that the first $n/2$ rows of T have sum $(n - 2)/2$, and that the remaining rows have sum $n/2$. Consider the right Perron vector for T , partitioned as $\begin{bmatrix} u \\ v \end{bmatrix}$, where u and v are vectors of order $n/2$, normalized so that $\mathbf{1}^t u + \mathbf{1}^t v = n$. From our development at the beginning of this section, we see that u and v satisfy (5)–(9). Further, pre- and post-multiplying (1) by the right Perron vector yields

$$\begin{aligned}
(\mathbf{1}^t u + \mathbf{1}^t v)^2 &= n^2 = (2\rho + 1)(u^t u + v^t v) \\
&= (n - 2\gamma/n + \mathcal{O}(1/n^2))(u^t u + v^t v).
\end{aligned}$$

Next, we write u as $u = u_{\max} \mathbf{1} - x$ and v as $v = v_{\min} \mathbf{1} + y$; note that x and y satisfy the constraints (10)–(13). In particular, $\mathcal{A}(\mu, v, \rho) \neq \emptyset$. We find that

$$\begin{aligned} (u^t u + v^t v) &= \frac{n}{2}(u_{\max}^2 + v_{\min}^2) + 2(v_{\min} \mathbf{1}^t y - u_{\max} \mathbf{1}^t x) + x^t x + y^t y \\ &= \frac{n}{2}(u_{\max}^2 + v_{\min}^2) + 2(n/\rho)(n(n-2)/4 - \rho^2) \\ &\quad \times \left(v_{\min} - \frac{\rho+1}{\rho} u_{\max} \right) + x^t x + y^t y. \end{aligned}$$

Applying parts (a) and (b) of Lemma 2.5, as well as Corollary 2.4, it follows that

$$\begin{aligned} u^t u + v^t v &\geq n + \frac{4\gamma^2 + 4\gamma - 1}{n} - \frac{16\gamma^2 - 4\gamma}{n} + \frac{4(4\gamma - 1)^3}{(27\gamma - 9)n} + \mathcal{O}(1/n^2) \\ &= n + \frac{4\gamma^2}{n} + \frac{(4\gamma - 1)^2(5 - 11\gamma)}{(27\gamma - 9)n} + \mathcal{O}(1/n^2). \end{aligned}$$

Consequently,

$$\begin{aligned} n^2 &= \left(n - \frac{2\gamma}{n} + \mathcal{O}(1/n^2) \right) (u^t u + v^t v) \\ &\geq \left(n - \frac{2\gamma}{n} \right) \left(n + \frac{4\gamma^2}{n} + \frac{(4\gamma - 1)^2(5 - 11\gamma)}{(27\gamma - 9)n} \right) + \mathcal{O}(1/n) \\ &= n^2 + 4\gamma^2 + \frac{(4\gamma - 1)^2(5 - 11\gamma)}{(27\gamma - 9)n} - 2\gamma + \mathcal{O}(1/n). \end{aligned}$$

It now follows that for n sufficiently large, $(4\gamma^2 - 2\gamma)(27\gamma - 9) + (4\gamma - 1)^2(5 - 11\gamma) \leq 0$, or equivalently,

$$-68\gamma^3 + 78\gamma^2 - 33\gamma + 5 \leq 0. \quad (19)$$

The cubic on the left-hand side of (19) has just one real root, namely

$$\frac{2(3^{2/3}) - 3^{4/3} + 13}{34},$$

and the lower bound on γ now follows. \square

The following is immediate.

Corollary 2.7. *For all sufficiently large even n , a tournament matrix T which maximizes the Perron value satisfies*

$$\rho(T) = \frac{n-1}{2} - \frac{\gamma_n}{n} + \mathcal{O}(1/n^2),$$

where

$$\frac{2(3^{2/3}) - 3^{4/3} + 13}{34} \leq \gamma_n \leq \frac{e^2 - 1}{2(e^2 + 1)}.$$

Remark 2.8. Theorem 2.6 provides an improvement on Friedland's bound (2), and it does so (essentially) by minimizing $u^t u + v^t v$ subject to the constraints (5)–(9). We note that if we drop constraints (8) and (9), and minimize $u^t u + v^t v$ subject only to (5)–(7), then we can recover Friedland's original bound $\gamma \geq \frac{3}{8}$ along the following lines.

Arguments analogous to those in Lemma 2.1 and Corollary 2.2 establish that the minimum possible value for $u^t u + v^t v$ will be attained when $\mu = \nu = \tau$. Under that hypothesis, an application of Kuhn–Tucker theory shows that the minimizing choice of u and v occurs when there are constants c_1, c_2, c_3 such that

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\tau \\ 0 & 1 & \tau \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Applying (5)–(9) and solving then produces closed form expressions for c_1, c_2 and c_3 in terms of n and ρ , from which we find that the minimum possible value of $u^t u + v^t v$ is given by

$$\begin{aligned} f(n, \rho) \equiv & 2n(2\rho^2 - 2(n-1)\rho + (n^2 - 2n + 2)/2) \\ & + 48n(\rho^2 - (n-2)\rho/2 - (n-2)/4)^2/(n^2 - 4). \end{aligned}$$

Arguing as at the end of the proof of Theorem 2.6 yields the inequality $n^2 = (2\rho + 1)(u^t u + v^t v) \geq (2\rho + 1)f(n, \rho)$, which then leads to

$$\begin{aligned} & \{\rho^2 - (n-2)\rho/2 - (n-2)/4\}\{(n-1-2\rho)(n^2 - 4) \\ & - 12(2\rho + 1)(\rho^2 - (n-2)\rho/2 - (n-2)/4)\} \geq 0. \end{aligned}$$

It is shown in Corollary 1.4 of [7] that the first factor above is nonnegative, and so the bound (2) follows readily. In particular, this line of reasoning does not require the additional hypothesis that $\gamma > 1/3$ that was needed in some of the results in the present paper; careful inspection of the proof of Theorem 2.3 shows that the need for that additional hypothesis arises from the nonnegativity constraints on x and y , which are, of course, equivalent to (8) and (9).

Remark 2.9. The distance between the upper and lower bounds on γ_n arising from (3) is approximately 0.005797, while the distance between the bounds arising from Corollary 2.7 is approximately 0.003343, representing an improvement of roughly 42.3%.

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